SOME APPROXIMATE SOLUTIONS OF STEFAN'S PROBLEM

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We obtain approximate solutions of Stefan's problem for various boundary conditions and for an initial temperature equal to the phase change temperature; the distribution of temperature along a coordinate direction in a region of increasing phase is given in the form of a quadratic expression.

For practical purposes approximate solutions of Stefan's problem are usually obtained assuming a linear temperature distribution in a region of increasing phase [3, 4]. The classical solutions of Lamé – Clapeyron and Stefan (1) are only suitable for a boundary condition of the first kind. We seek a solution for the temperature of an increasing phase in the trinormal form

$$T = Ax^2 + Bx + C, \tag{1}$$

where the function C(t) has the physical meaning of temperature of the exterior surface of the body. We assume this surface to be planar and to contain the coordinate origin; the OX axis is taken in a direction along the interior normal to this surface. The initial temperature, and hence also the temperature at all points of decreasing phase, at an arbitrary time instant are assumed to be equal to the phase transition temperature, which we take as our origin for temperature calculations.

Determining the solutions of nonlinear problems of heat and mass transfer in the form (1) usually proves to be sufficiently accurate for practical purposes ([5], et al.). The initial conditions we assume correspond, to one degree or another, to that for cast hardening, to the deepening vaporization zone in the drying of a capillary-porous body [2], and to other thermophysical processes.

The problem in question may be described as follows:

$$a\frac{\partial^2 T(x,\ t)}{\partial x^2} = \frac{\partial T(x,\ t)}{\partial t} \qquad (0 \leqslant x \leqslant \xi), \tag{2}$$

$$T(x,\ 0) = 0, \tag{3}$$

$$T(x,\ t) = 0 \qquad (x \geqslant \xi), \tag{4}$$

$$-\lambda\frac{\partial T(\xi,\ t)}{\partial x} = \varkappa\frac{d\xi}{dt}. \tag{5}$$

In writing the Stefan condition (5) we assume that $\varkappa > 0$ for absorption and that $\varkappa < 0$ when heat is liberated during a phase transition. The Eqs. (2)-(5) must be supplemented by a boundary condition relating the body and its surrounding medium. The Eqs. (3) and (4) superimpose the condition

$$\xi(0) = 0 \tag{5a}$$

on the motion of the boundary separating the phases.

The coefficients A, B, and C must be chosen so that at $x = \xi$ the Eq. (2) becomes an identity, i.e., as is evident from Eqs. (1) and (2), the following equation must hold for the unknown functions of the time:

$$2aA = \xi^2(dA/dt) + \xi(dB/dt) + dC/dt. \tag{6}$$

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For the approximation (1) the conditions (5) and (4) assume the form

$$\lambda(2A\xi + B) = -\varkappa \frac{d\xi}{dt}, \tag{7}$$

$$A\xi^2 + B\xi + C = 0. \tag{8}$$

On the surface x = 0 of the body we are given a boundary condition of the third kind:

$$-\lambda\left[\partial T(0,\ t)/\partial x\right] = \alpha(P - C),$$

which, for the approximation (1), may be written as

$$B = \alpha\lambda^{-1}(C - P). \tag{9}$$

From Eqs. (7), (8), and (9) we find

$$A = (2\lambda + \alpha\xi)^{-1}\left(\frac{\alpha P}{\xi} - \frac{\varkappa}{\xi}\cdot\frac{d\xi}{dt} - \frac{\alpha\varkappa}{\lambda}\cdot\frac{d\xi}{dt}\right), \tag{10}$$

$$C = (2\lambda + \alpha\xi)^{-1}\left(\alpha P\xi + \varkappa\xi\frac{d\xi}{dt}\right). \tag{11}$$

Upon substituting the Eqs. (9), (10), and (11) into Eq. (6), we obtain, after involved but elementary manipulations, a differential equation for the function $\xi(t)$:

$$\frac{\varkappa}{\lambda}\left(\frac{d\xi}{dt}\right)^2 = 2a(2\lambda + \alpha\xi)^{-1}\left[\frac{\alpha P}{\xi} - \left(\frac{\varkappa}{\xi} + \frac{\alpha\varkappa}{\lambda}\right)\frac{d\xi}{dt}\right]. \tag{12}$$

In solving this equation we consider four cases. In the first case we neglect the volumetric heat capacity of the body, i.e., we let $a \to \infty$. Integrating the equation obtained as a result of passing to the limit in Eq. (12), we obtain, subject to the condition (5a), the solution

$$\xi = -\frac{\lambda}{\alpha} + \sqrt{\left(\frac{\lambda}{\alpha}\right)^2 + \frac{2\lambda P}{\varkappa}t}, \tag{13}$$

which, in dimensionless quantities, takes the form

$$\eta = -1 + (1 + 2\Lambda\tau)^{1/2}.$$

In the second case we neglect the heat capacity and assume that a boundary condition of the first kind relates the body with the surrounding medium, i.e., we let $a \to \infty$, $\alpha \to \infty$.

The solution (13) is then transformed into the equation

$$\xi = (2\lambda\varkappa^{-1}Pt)^{1/2}, \quad \text{or} \quad \xi = w\cdot 2(at)^{1/2},$$

where the dimensionless coefficient of proportionality is given by

$$w = (0.5\Lambda)^{1/2}. \tag{14}$$

In the third case we consider solving our problem by using the approximation (1), but with a boundary condition of the first kind, i.e., for $\alpha \to \infty$. As a result of the corresponding passage to the limit, we obtain from Eqs. (9), (10), and (11)

$$A = \frac{P}{\xi^2} - \frac{\varkappa}{\lambda\xi}\cdot\frac{d\xi}{dt}, \tag{15}$$

$$B = \frac{\varkappa}{\lambda}\cdot\frac{d\xi}{dt} - \frac{2P}{\xi}, \tag{16}$$

$$C = P. \tag{17}$$

As a result of integrating the equation obtained by letting $\alpha \to \infty$ in Eq. (12), we obtain, subject to the condition (5a), the solution

$$\xi = \sqrt{2\left(-a + \sqrt{a^2 + \frac{2a\lambda P}{\varkappa}}\right)t}. \tag{18}$$

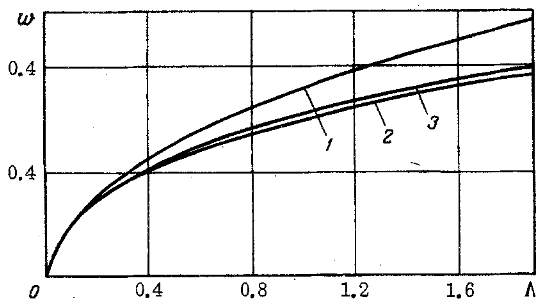

Fig. 1. Relationship between the values of w and Λ: 1) by Eq. (14); 2) by Eq. (19); 3) in accord with the Lamé–Clapeyron solution [1].

If in Eq. (18) we divide the coefficient of $\sqrt{t}$ by $2\sqrt{a}$, we obtain

$$w = \left[0.5\left(\sqrt{1+2\Lambda}-1\right)\right]^{1/2}. \tag{19}$$

In the figure we show the graph of $w = w(\Lambda)$, calculated in accord with Eqs. (14) and (19), and also the graph obtained as the result of graphically solving the transcendental equation derived from the Lamé –Clapeyron solution ([1], p. 424). In comparing both graphs it is evident that for determining the law of motion of the phase separation boundary our approximate solution is no worse than that obtained by the Lamé –Clapeyron method.

Finally, we consider the fourth case, the solution of Eq. (12) in its general form. Writing Eq. (12) in dimensionless variables, we find

$$\frac{d\eta}{d\tau} = \frac{1+\eta}{\eta(2+\eta)}\left[\left(1+2\Lambda\frac{\eta(2+\eta)}{(1+\eta)^2}\right)^{1/2}-1\right]. \tag{20}$$

We make the change of variables

$$u = \sqrt{1+2\Lambda\frac{\eta(2+\eta)}{(1+\eta)^2}}$$

and, noting that when $\tau = 0$ the new variable, by virtue of the condition (5a), is equal to 1, we find the integral of Eq. (20) to be

$$\tau = \frac{1}{2}\left[\frac{u+1}{1+2\Lambda-u^2} - \frac{1}{2\sqrt{1+2\Lambda}}\ln\frac{\left(\sqrt{1+2\Lambda}+u\right)\left(\sqrt{1+2\Lambda}-1\right)}{\left(\sqrt{1+2\Lambda}-u\right)\left(\sqrt{1+2\Lambda}+1\right)} - \frac{1}{\Lambda}\right]. \tag{21}$$

We return now to the boundary condition of the second kind, i.e.,

$$-\lambda\left[\partial T(0,\ t)/\partial x\right] = Q.$$

In the expression (1), obviously,

$$B = -\frac{Q}{\lambda}. \tag{22}$$

Adjoining the expression (22) to the Eqs. (6)-(9) and solving the resulting system, we obtain

$$A = \frac{1}{2\lambda\xi}\left(Q - \varkappa\frac{d\xi}{dt}\right), \tag{23}$$

$$C = \frac{\xi}{2\lambda}\left(\varkappa\frac{d\xi}{dt} + Q\right), \tag{24}$$

$$\frac{d\xi}{dt} = \frac{a}{2\xi}\left[\left(1+\frac{4Q\xi}{a\varkappa}\right)^{1/2} - 1\right]. \tag{25}$$

In Eq. (25) we now change over to the dimensionless quantities ζ and θ, and we then integrate the equation with the aid of the substitution $z = \sqrt{1+4\zeta}$. Upon reverting to the variables ζ and θ, we may write the integral of this equation as follows:

$$24\theta = \left[2(1+4\zeta)^{3/2} + 3(1+4\zeta) - 5\right]. \tag{26}$$

When the volumetric heat capacity is small, i.e., when the quantity a is large, the dimensionless coordinate $\zeta \ll 1$. Replacing, on the basis of this condition, the right side of Eq. (26) by its approximate expression, we obtain

$$\zeta = \theta, \quad \text{or} \quad \xi = Q\varkappa^{-1}t.$$

In view of the conditions (3) and (4), the solutions we have obtained are suitable for both a semi-bounded body and for an unbounded plate when these latter interact symmetrically with the surrounding medium. The solutions for a sphere and for a long cylinder can be obtained by replacing Eq. (2) by the equation for the corresponding body. Solutions can be obtained for exterior boundary conditions which are monotone functions of the time and also when the thermophysical coefficients of the increasing phase depend linearly on the temperature.

NOTATION

t	is the time;
x	is the coordinate;
$\xi = \xi(t)$	is the coordinate of phase interface;
T(x, t)	is the temperature of body;
a and λ	are the thermal diffusivity and thermal conductivity of growing phase;
$\varkappa$	is the volumetric heat of change of phase;
P = const	is the temperature of medium (at boundary conditions of the first and third kinds);
Q = const	is the density of heat flux at the body–medium interface (at boundary condition of the second kind);
A, B, and C	are the time-dependent coefficients;
α	is the heat-transfer coefficient;
$\Lambda = \lambda P/a\varkappa$, $\eta = a\xi/\lambda$, $\tau = a\alpha^2 t/\lambda^2$, $\zeta = Q\xi/a\varkappa$, $\theta = Q^2 t/a\varkappa^2$	are the dimensionless characteristics of the process

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